Journal of Statistical Physics, Vol. 67, Nos. 5/6, 1992

Remarks on Scaling a Model of Witten–Sander Type

Peter March[1]

Received February 26, 1992; final February 18, 1992

We try to prove rigorously that the perimeter of the large Witten–Sander cluster does not scale as the square root of its area, by making a forced comparison with the ill-posed Hele-Shaw problem of fluid dynamics. The attempt is not completely successful; nevertheless some interesting consequences of the comparison are derived.

KEY WORDS: Diffusion-limited aggregation; harmonic measure; Hele-Shaw problem; martingale problem.

1. INTRODUCTION

The Witten–Sander model, or diffusion-limited aggregation as it is usually called, is an enormously appealing growth process defined on $\mathbb{Z}^2$ via the prescription

$$P(c_{n+1}\backslash c_n = \{y\} \mid c_n = \gamma) = \lim_{|x| \to \infty} P_x(S(\tau_\gamma) = y) \tag{1.1}$$

In this formula the c_k are finite subsets of vertices of $\mathbb{Z}^2$, S is simple random walk, and τ_γ is the first time S encounters a nearest neighbor of the subset γ. If we set $c_1 = \{0\}$, then c_n is a random, connected cluster of n lattice sites which contains the origin, and one would like to describe the shape of a typical cluster for large values of n. It is expected that these large clusters are highly ramified and of low density, having diameter of greater order of magnitude than the square root of the number of vertices. These expectations are supported by physical theory and a lot of computer simulation. The interested reader may consult the original paper[27] as well as collections[21,22] and a survey.[26]

Not surprisingly, there is a dearth of rigorous statements about the behavior of large clusters. A notable exception is the theorem of

---

[1] Department of Mathematics, Ohio State University, Columbus, Ohio 43210.

0022-4715/92/0600-1117$06.50/0

Kesten[11,12] to the effect that if r_n is the radius of cluster c_n, then there is a fixed finite constant c such that

$$P(\limsup_{n \to \infty} n^{-2/3} r_n \leqslant c) = 1 \tag{1.2}$$

That is to say, if c_n does consist of a system of long arms and projections, these cannot grow in length faster than the two-thirds power of the number of vertices. The proof of (1.2) is based on an analogue, for the potential theory of $\mathbb{Z}^2$, of Beurling's circular projection theorem. It yields an upper bound on the hitting probabilities of clusters γ which appear in (1.1), the bound depending only on the radius of γ and being otherwise independent of its shape. [It should be mentioned that this growth process can be defined on $\mathbb{Z}^d$, $d \geqslant 3$, and Kesten[12] proves upper bounds on r_n similar to (1.2) but with dimension-dependent rate $n^{-2/d}$.]

The question of lower bounds on r_n is most attractive, but it would seem to depend on shape-independent lower bounds on hitting probabilities. Readers can convince themselves that there are no nontrivial such bounds. Even shape-dependent lower bounds seem so formidable as to be quite discouraging. So, to make a statement to the effect that the linear size of the large cluster really is of greater order of magnitude than the square root of its area we must take another tack.

An early observation about the Witten–Sander model is that it seems to be a discrete, stochastic version of the Hele-Shaw problem from the theory of two-phase flow.[18] Physically speaking, the Hele-Shaw problem is to describe the time development of a bubble of air in water between two parallel plates under an applied suction from the perimeter of the plates which is taken to be very far from the bubble. If the gap between the plates is so small that the problem is treated as two-dimensional and the pressure inside the air bubble is taken as constant, the Hele-Shaw cell can be approximated by a moving boundary problem for Ω_t, the unbounded region of the plane occupied by water. If $v(x, t)$ is the fluid velocity in Ω_t and $u(x, t)$ is the pressure, then, assuming the fluid is incompressible,

$$\begin{aligned} v(x, t) &= \lambda \nabla u(x, t) \qquad \text{in} \quad \Omega_t \\ v(x, t) \cdot n_{\Omega_t}(x) &= \lambda \frac{\partial}{\partial n_{\Omega_t}} u(x, t) \qquad \text{on} \quad \partial\Omega_t \end{aligned} \tag{1.3a}$$

and

$$\begin{aligned} \Delta u(x, t) &= 0 \qquad \text{in} \quad \Omega_t \\ u(x, t) &= \sigma \kappa_{\Omega_t}(x) \qquad \text{on} \quad \partial\Omega_t \\ u(x, t) &\sim \frac{1}{2\pi} \log |x| \qquad \text{as} \quad |x| \to \infty \end{aligned} \tag{1.3b}$$

Here n_{Ω_t} is the outward normal vector field of $\partial\Omega_t$, κ_{Ω_t} is the curvature, σ is surface tension, and $\lambda = b^2/\mu$, where b is the gap between the plates and μ is the fluid viscosity.[1] If we set $\lambda = 1$ and $\sigma = 0$, then $u(x, t)$ is the Green function for Ω_t with pole at infinity and so

$$h(\Omega_t)(dx) \equiv \frac{\partial}{\partial n_{\Omega_t}} u(x, t)\, \sigma_{\Omega_t}(dx)$$

is the corresponding Poisson kernel. Of course, $h(\Omega_t)$ has a probabilistic interpretation as the first hitting distribution on $\partial\Omega_t$ of a Brownian particle released at $+\infty$. So it is in this case that Ω_t^c, the region occupied by air, can be thought of as the continuum version of the Witten–Sander cluster c_n.

To place these two problems on an equal footing, consider a sequence of Witten–Sander models on the scaled lattice $(1/N)\,\mathbb{Z}^2$ and replace each finite cluster $c_N(n) \subset (1/N)\,\mathbb{Z}^2$ with the closed set

$$C_N(n) = \bigcup_{y \in c_N(n)} E_N(y); \qquad E_N(y) = y + \left[-\frac{1}{2N}, \frac{1}{2N}\right]^2 \tag{1.4}$$

Then we have a sequence of Markov chains with transition probabilities

$$\text{Prob}(C_N \to C_N \cup E_N(y)) = h_N^*(C_N)(y) \tag{1.5}$$

where $h_N^*(C_N)$ is a probability measure supported by the nearest neighbors in $(1/N)\,\mathbb{Z}^2$ of C_N. For the Witten–Sander model $h_N^* = h_N^{\mathrm{RW}}$, the hitting distribution of random walk from infinity, as in (1.1). One could consider other choices, for example,

$$h_N^*(C_N)(y) = \begin{cases} \int_{\partial C_N \cap E_N(y)} h(C_N^c)(dz), & y \in C_N^c \cap (1/N)\,\mathbb{Z}^2 \\ 0, & \text{otherwise} \end{cases} \tag{1.6}$$

which is the probability that a Brownian particle, rather than a random walker, hits C_N near to y. If, as a first step in a passage to the limit, we consider the possible states C_N as measures $1_{C_N}(x)\,dx$, then since each elementary transition $C_N \to C_N \cup E_N(y)$ adds area $1/N^2$ to the current state of the chain, we should permit N^2 such transitions per unit of real time to force a unit increase in area per unit time. This is the *law of large numbers* scaling for cluster area and is a necessary condition for comparison of these chains with the Hele-Shaw problem. However, to *force* a passage to this limit we will require the continuity of the map

$$\mu(dx) \to h(spt\ \mu)(dx) \tag{1.7}$$

where μ is a measure on $\mathbb{R}^2$, $spt\,\mu$ is its closed support, and $h(spt\,\mu)$ is harmonic measure of $spt\,\mu$ as seen from infinity. Unfortunately, this assignment is not continuous, in general. Consider, for example the sets

$$U = \{x: |x| \leqslant 1\}, \qquad V = \{(x^1, 0): 0 \leqslant x^1 \leqslant 2\}$$

and

$$V_l = \{(x^1, x^2): 0 \leqslant x^1 \leqslant 2, |x^2| \leqslant 1/l\} \tag{1.8}$$

Clearly $1_{U \cup V_l}(x)\,dx$ converges to $1_U(x)\,dx$, while $h(U \cup V_l)(dx)$ converges to $h(U \cup V)(dx)$, which is distinct from $h(U)(dx)$, since the needle V has nonzero length. However, Theorem 2.5 below gives a sufficient condition in terms of cluster *perimeter* for such continuity. Thus, if we wish to pick up the Hele-Shaw problem in the limit, we are obliged to track not only cluster area, but also cluster perimeter. It is important to note, however, that from the point of view of perimeter the scaling above is the *central limit* scaling. (We make a further remark on this point in the last section.)

Now the zero-surface-tension Hele-Shaw problem is ill-posed: it has a weak formulation as an nonlinear backward heat equation as follows. (See also Remark 3.5 below.) By integrating a time-dependent, rapidly vanishing test function ϕ over Ω_t and using Eqs. (1.3a) and (1.3b) we obtain, in the case $\lambda = 1$ and $\sigma = 0$,

$$\frac{\partial}{\partial t}\int \phi(x, t)\, 1_{\Omega_t}(x)\, dx = \int \frac{\partial}{\partial t}\phi(x, t)\, 1_{\Omega_t}(x)\, dx - \int \phi(x, t)\, h(\Omega_t)(dx) \tag{1.9}$$

Now since ϕ vanishes at infinity, the Riesz decomposition for potentials (or Itô's formula) gives, for each t,

$$\int \phi(x, t)\, h(\Omega_t)(dx) = \int \Delta\phi(x, t)\, u(x, t)\, dx \tag{1.10}$$

Furthermore, if each Ω_t is open and connected, then

$$1_{\Omega_t}(x)\, dx = H(u(x, t))\, dx \tag{1.11}$$

where

$$H(u) = \begin{cases} 1, & u > 0 \\ 0, & u \leqslant 0 \end{cases}$$

Hence, on integrating (1.9) from 0 to t, we obtain

$$\begin{aligned} &\langle \phi(\cdot, t), H(u(\cdot, t)) \rangle - \langle \phi(\cdot, 0), H(u(\cdot, 0)) \rangle - \int_0^t \left\langle \frac{\partial}{\partial s}\phi(\cdot, s), H(u(\cdot, s)) \right\rangle ds \\ &\quad = -\int_0^t \langle \Delta\phi(\cdot, s), u(\cdot, s) \rangle\, ds \end{aligned} \tag{1.12}$$

or in other words

$$\frac{\partial}{\partial t} H(u(x, t)) = -\Delta u(x, t) \tag{1.13}$$

Equation (1.13) is not expected to have solutions for arbitrary initial domains. Thus one surmises that the ramified shape of large Witten–Sander clusters may be due to the fact that the complements of these clusters are trying to approximate an ill-posed moving boundary problem.

Now the main idea of the paper is just this: *assume* that the perimeter process of an appropriately scaled sequence of Witten–Sander models on $(1/N)\,\mathbb{Z}^2$ is well enough behaved so that the corresponding sequence of martingale problems can be proved to pass to the limit, yielding a solution to an ill-posed problem. If the limit of the initial clusters is not of a special class, this leads to a contradiction. The contradiction then proves that the scaled models in fact are poorly behaved and a qualitative statement about large clusters is obtained.

Unfortunately, a technical point intervenes to prevent the complete success of this gambit: the limit martingale problem is in general a weak form of the ill-posed Hele-Shaw cell as in (1.12) *plus* an extra term having to do with the possibility of condensation of fluid bubbles (see Theorem 3.4).

Thus, it develops, in a way made precise in Theorem 3.8, that for the sequence of scaled Witten–Sander-type models, either (i) the cluster perimeter is badly behaved, or (ii) the perimeter is well behaved, but, in the limit, bubbles condense immediately.

Our approach is open to at least two criticisms. First, it is not at all apparent that cluster perimeter is the correct measure of linear size with which to work. Its use is determined by our method rather than by its being *a priori* an intrinsically useful quantity in terms of which to estimate the growth of cluster size. Second, we cannot determine whether or not bubbles do condense immediately. Deciding the question would seem to require some definite estimates of discrete harmonic measure and these are not forthcoming, at least from this author.

The next section contains a result on the continuity of harmonic measure as a function of domain, which is used to pass to the continuum limit. Section three concerns itself with martingale problems and the possible behavior of solutions of the ill-posed Hele-Shaw problem. The final section includes a brief discussion of models incorporating a form of surface tension and ends with some further questions.

2. HARMONIC MEASURE

The point of this section is to introduce the terms with which we discuss passage to the continuum limit. At the Nth step we consider a growth process on $(1/N)\,\mathbb{Z}^2$, the state of which is a finite connected cluster of sites c_N (connectedness in the sense of lattice paths from site to site) or equivalently a piecewise C^1 domain

$$C_N = \bigcup_{y \in c_N} E_N(y); \qquad E_N(y) = y + \left[-\frac{1}{2N}, \frac{1}{2N}\right]^2 \tag{2.1}$$

Definition 2.1. (a) A *grid-square domain* is any finite union of squares of the form (2.1); (b) If C is a piecewise C^1 domain, that is, ∂C is a finite union of C^1 curves of finite arc length, then the *boundary measure* of C is the measure on $S^1 \times \mathbb{R}^2$ defined by

$$V_C(d\xi\, dx) = [|n^1(x)|\, \delta_{\operatorname{sgn}(n^1(x))\, e_1}(d\xi) + |n^2(x)|\, \delta_{\operatorname{sgn}(n^2(x))\, e_2}(d\xi)] \otimes \sigma_C(dx)$$

where $n = (n^1, n^2)$ is the unit normal to ∂C at x, and σ_C is the arc length measure on ∂C, the e_i are standard basis vectors, and sgn is algebraic sign; (c) the *perimeter measure* of C, denoted $\bar{V}_C(dx)$, is the marginal of V_C on $\mathbb{R}^2$, that is, for bounded continuous functions ϕ,

$$\int \phi(x)\, \bar{V}_C(dx) = \iint 1 \otimes \phi(\xi, x)\, V_C(d\xi\, dx)$$

Warning. In general $V_C(d\xi\, dx) \neq \delta_{n_C(x)}(d\xi) \otimes \sigma_C(dx)$ and $\bar{V}_C(dx) \neq \sigma_C(dx)$ unless C is a grid-square domain.

Let us denote by $\mathscr{V}$ the closure, in the space of Borel measures on $S^1 \times \mathbb{R}^2$ under weak convergence, of the set of boundary measures of grid-square domains determined by all finite *connected* clusters $c_N \subset (1/N)\,\mathbb{Z}^2$ for all $N \geqslant 1$; and by $\bar{\mathscr{V}}$ the corresponding closure of perimeter measures.

Elements of $\mathscr{V}$ are, essentially, examples of the *varifolds* of geometric measure theory. They keep track not only of perimeter, but also of the oriented tangent line. When smooth domains are approximated in area by grid-squares, the tangent line and perimeter can be approximated only in a very weak sense, which is made precise in the next proposition. Boundary measures play an important but intermediate role in the proof of Theorem 2.5.

Proposition 2.2. Let C be a piecewise C^1 domain. There is a sequence of grid-square domains C_N, $N \geqslant 1$, such that (i) $\lim_{N \to \infty} 1_{C_N} = 1_C$ almost everywhere, and (ii) $\lim_{N \to \infty} V_{C_N}(d\xi\, dx) = V_C(d\xi\, dx)$.

Proof. Our candidate is

$$C_N = \bigcup_{y \in C \cap (1/N)\mathbb{Z}^2} E_N(y)$$

and certainly (i) is fulfilled by this choice. Suppose first that ∂C is a C^1 curve. Since $n(\cdot) \equiv n_C(\cdot)$ is uniformly continuous, given $\varepsilon > 0$ there is $0 < \delta \leqslant \varepsilon$ such that if $\{z_k : k \leqslant M(\delta)\}$ is a partition of ∂C of mesh less than δ and $J_{\varepsilon,k}$ are the subcurves of ∂C determined by the partition, then for any k and $x, y \in J_{\varepsilon,k}$,

$$|n(x) - n(y)| \leqslant \varepsilon \tag{2.2}$$

Let $L_{\varepsilon,k}$ be the segment with endpoints z_{k-1} and z_k and let C_ε be the corresponding polygonal approximation to C. By the mean value theorem, each $L_{\varepsilon,k}$ is parallel to the tangent line to $J_{\varepsilon,k}$ at some point between z_{k-1} and z_k, so that if $n_{\varepsilon,k}$ denotes the constant normal vector to $L_{\varepsilon,k}$, then

$$|n(\cdot) - n_{\varepsilon,k}| \leqslant \varepsilon \qquad \text{on} \quad J_{\varepsilon,k} \tag{2.3}$$

Now notice that the assignment

$$\eta \to |\eta^1|\, \delta_{\operatorname{sgn}(\eta^1)\, e_1} + |\eta^2|\, \delta_{\operatorname{sgn}(\eta^2)\, e_2} \tag{2.4}$$

from $\eta = (\eta^1, \eta^2) \in S^1$ to measures on S^1 is weakly continuous. Since, by (2.3), n_{C_ε} converges uniformly to n_C and $\sigma_{C_\varepsilon}(dx)$ converges weakly to $\sigma_C(dx)$, it follows that for any bounded, continuous test function $f: S^1 \times \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_{\varepsilon \to 0} |\langle f, V_{C_\varepsilon}\rangle - \langle f, V_C\rangle| = 0 \tag{2.5}$$

(Here we use $\langle \phi, \mu\rangle$ to denote the integral of a function ϕ against a measure μ.)

Now let us compare V_{C_ε} with V_{C_N} when $1/N \ll \varepsilon$. Let $w_{N,k}$ be a corner point of ∂C_N nearest z_k and let $I_{N,k}$ be that part of ∂C_N with endpoints $w_{N,k-1}$ and $w_{N,k}$. By (2.3), $I_{N,k}$ consists of at most $N(|n^1_{\varepsilon,k}| + \varepsilon)\, |J_{\varepsilon,k}| + 2$ vertical segments of length $1/N$ and at most $N(|n^2_{\varepsilon,k}| + \varepsilon)\, |J_{\varepsilon,k}| + 2$ horizontal segments of length $1/N$. (A glance at Fig. 1 may help.) Since the endpoints of $I_{N,k}$ are also within distance $1/N$ of the endpoints of $L_{\varepsilon,k}$, at least $N\, |n^1_{\varepsilon,k}|\, |L_{\varepsilon,k}| - 2$ of these segments have normal vector $\operatorname{sgn}(n^1_{\varepsilon,k})\, e_1$ and at least $N\, |n^2_{\varepsilon,k}|\, |L_{\varepsilon,k}| - 2$ of these segments have normal vector $\operatorname{sgn}(n^2_{\varepsilon,k})\, e_2$. Observing that there is a fixed constant K such that

$$J_{\varepsilon,k} \subset T_{\varepsilon,k} = \{y: \operatorname{dist}(y, L_{\varepsilon,k}) \leqslant K\varepsilon\} \tag{2.6}$$

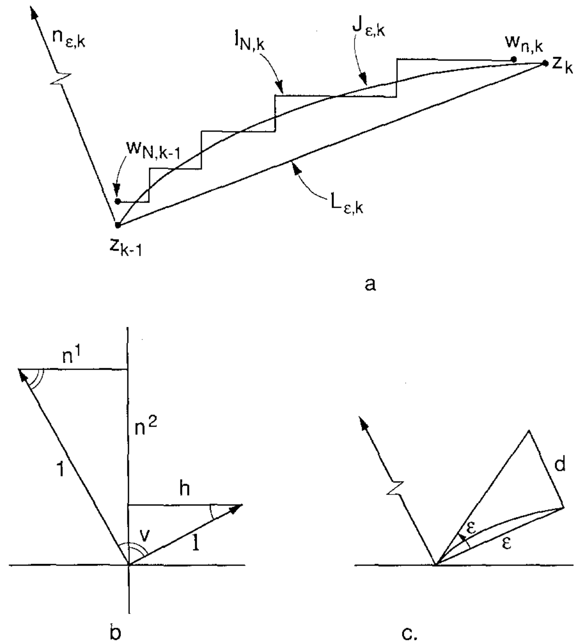

Fig. 1. (a) Diagram to aid in the proof of Proposition 2.2. (b) $v = n^1 l$ and $h = n^2 l$. (c) $d = \varepsilon \tan \varepsilon \leqslant K\varepsilon$.

we have for nonnegative, bounded continuous functions the estimates

$$\int_{I_{N,k}} f(n_{C_N}(x), x)\, \sigma_{C_N}(dx)$$
$$\leqslant \frac{2\,\|f\|_\infty}{N} + |J_{\varepsilon,k}| \{ (|n^1_{\varepsilon,k}| + \varepsilon) \sup_{x \in T_{\varepsilon,k}} f(\operatorname{sgn}(n^1_{\varepsilon,k})\, e_1, x)$$
$$+ (|n^2_{\varepsilon,k}| + \varepsilon) \sup_{x \in T_{\varepsilon,k}} f(\operatorname{sgn}(n^2_{\varepsilon,k})\, e_2, x) \} \tag{2.7a}$$

$$\int_{I_{N,k}} f(n_{C_N}(x), x)\, \sigma_{C_N}(dx)$$
$$\geqslant \frac{-2\,\|f\|_\infty}{N} + |L_{\varepsilon,k}| \{ |n^1_{\varepsilon,k}| \inf_{x \in T_{\varepsilon,k}} f(\operatorname{sgn}(n^1_{\varepsilon,k})\, e_1, x)$$
$$+ |n^2_{\varepsilon,k}| \inf_{x \in T_{\varepsilon,k}} f(\operatorname{sgn}(n^2_{\varepsilon,k})\, e_2, x) \} \tag{2.7b}$$

Now, since

$$\langle f, V_{L_{\varepsilon,k}} \rangle = \int_{L_{\varepsilon,k}} \sum_{i=1}^{2} [|n^i_{\varepsilon,k}|\, f(\operatorname{sgn}(n^i_{\varepsilon,k})\, e_i, x)]\, \sigma(dx) \tag{2.8}$$

$\langle f, V_{L_{\varepsilon,k}}\rangle$ also satisfies the bounds in (2.7a) and (2.7b). Using the notation

$$\operatorname{osc}(f, \varepsilon) = \max_{\xi = \pm e_i} \sup_x \Big[\sup_{|x-y| \leqslant \varepsilon} f(\xi, y) - \inf_{|x-y| \leqslant \varepsilon} f(\xi, y)\Big] \tag{2.9}$$

we have then

$$\begin{aligned}
&|\langle f, V_{C_N}\rangle - \langle f, V_{C_\varepsilon}\rangle| \\
&\quad \leqslant \sum_{k=1}^{M(\delta)} |\langle f, V_{I_{N,k}}\rangle - \langle f, V_{L_{\varepsilon,k}}\rangle| \\
&\quad \leqslant \frac{4\,\|f\|_\infty M(\delta)}{N} + \sum_{i=1}^{2} \sum_{k=1}^{M(\delta)} \Big[(|n^i_{\varepsilon,k}| + \varepsilon)\,|J_{\varepsilon,k}| \sup_{x \in T_{\varepsilon,k}} f(\operatorname{sgn}(n^i_{\varepsilon,k})\, e_i, x) \\
&\qquad - |n^i_{\varepsilon,k}|\,|L_{\varepsilon,k}| \inf_{x \in T_{\varepsilon,k}} f(\operatorname{sgn}(n^i_{\varepsilon,k})\, e_i, x)\Big] \\
&\quad \leqslant \frac{4\,\|f\|_\infty M(\delta)}{N} + \sum_{i=1}^{2} \sum_{k=1}^{M(\delta)} \big[(|n^i_{\varepsilon,k}| + \varepsilon)\,|J_{\varepsilon,k}| - |n^i_{\varepsilon,k}|\, L_{\varepsilon,k}|\big]\, \|f\|_\infty \\
&\qquad + \sum_{k=1}^{M(\delta)} |n^i_{\varepsilon,k}|\,|L_{\varepsilon,k}| \operatorname{osc}(f, K\varepsilon) \\
&\quad \leqslant \frac{4\,\|f\|_\infty M(\delta)}{N} + \sum_{i=1}^{2} \sum_{k=1}^{M(\delta)} \big[\varepsilon\,|J_{\varepsilon,k}| + |n^i_{\varepsilon,k}|(|J_{\varepsilon,k}| - |L_{\varepsilon,k}|)\big]\, \|f\|_\infty \\
&\qquad + \sum_{k=1}^{M(\delta)} |n^i_{\varepsilon,k}|\,|L_{\varepsilon,k}| \operatorname{osc}(f, K\varepsilon) \\
&\quad \leqslant \frac{4\,\|f\|_\infty M(\delta)}{N} + 2\varepsilon\,|\partial C| \cdot \|f\|_\infty + 2(|\partial C| - |\partial C_\varepsilon|)\, \|f\|_\infty \\
&\qquad + |\partial C_\varepsilon| \operatorname{osc}(f, K\varepsilon)
\end{aligned} \tag{2.10}$$

Thus

$$\overline{\lim_{\varepsilon \to 0}}\; \overline{\lim_{N \to \infty}}\; |\langle f, V_{C_N}\rangle - \langle f, V_{C_\varepsilon}\rangle| = 0$$

and this together with (2.5) yields the proof in case ∂C is a C^1 curve. In case ∂C is piecewise C^1, the argument is entirely similar. ∎

Proposition 2.3. Let C_N, $N \geqslant 1$, be a sequence of piecewise C^1 domains such that $\lim_{N \to \infty} \bar{V}_{C_N}(dx) = \sigma(dx)$ for some measure σ. Then, for some subsequence N' and some $V \in \mathscr{V}$,

$$\lim_{N' \to \infty} V_{C_{N'}}(d\xi\, dx) = V(d\xi\, dx) \qquad \text{and} \qquad \bar{V}(dx) = \sigma(dx)$$

Proof. Since S^1 is a compact set, the sequence V_{C_N}, $N \geqslant 1$, is weakly compact on $S^1 \times \mathbb{R}^2$ if and only if $\bar{V}_{C_N}$, $N \geqslant 1$, is weakly compact on $\mathbb{R}^2$. Consequently, since $\bar{V}_{C_N}$ actually converges, some subsequence of V_{C_N} converges to, say, V. But then for any bounded continuous function ϕ,

$$\begin{aligned} \langle \phi, \bar{V} \rangle &= \langle 1 \otimes \phi, V \rangle = \lim_{N' \to \infty} \langle 1 \otimes \phi, V_{C'_N} \rangle \\ &= \lim_{N' \to \infty} \langle \phi, \bar{V}_{C_{N'}} \rangle = \langle \phi, \sigma \rangle \quad \blacksquare \end{aligned} \tag{2.11}$$

For a compact set $C \subset \mathbb{R}^2$, write

$$C^c = A \cup B \tag{2.12}$$

where A is the unbounded, connected component of the complement of C (the active region) and B is the union of the bounded, connected components (the bubbles). If, subsequently, C appears as a function of some parameters, then A and B, similarly decorated, will always have the above relationship to C.

Definition 2.4. Let $D \subset \mathbb{R}^2$ be an open set and define (i) $g_D(x)$ to be the Green function of D with pole at infinity; (ii) $h_D(dx)$ to be the classical harmonic measure of ∂D as seen from infinity; and (iii) comp D to be the unbounded connected component of D, provided D contains the complement of a compact set.

Here is the main theorem of this section, which depends crucially on Caratheodory's kernel theorem of conformal mapping.[2]

Theorem 2.5. Let C_N, $N \geqslant 1$, be a sequence of connected grid-square domains in $\mathbb{R}^2$ and let $A_N = \text{comp } C_N^c$ be the unbounded component of C_N^c. Suppose that for some measure σ, $\sigma_{A_N}(dx)$ converges weakly to $\sigma(dx)$. Let $A = \text{comp}(spt\ \sigma)^c$. Then:

(i) $\lim_{N \to \infty} g_{A_N}(x) = g_A(x)$ pointwise on $\mathbb{R}^2$.

(ii) $\lim_{N \to \infty} h_{A_N}(dx) = h_A(dx)$ weakly.

Furthermore, suppose $1_{C_N^c}(x)\, dx$ converges vaguely to some measure μ. Then:

(iii) $\mu|_{\{g_A > 0\}}(dx) = H(g_A(x))\, dx$, where

$$H(u) = \begin{cases} 1, & u > 0 \\ 0, & u \leqslant 0 \end{cases}$$

is the Heaviside function.

Proof. Let us emphasize the fact that since C_N is connected, A_N is simply connected when viewed as a subset of the Riemann sphere $\hat{\mathbb{C}}$; thus, conformal mapping methods can be applied. We show that as sets, A_N, $N \geqslant 1$, converge in Caratheodory's sense to A. Caratheodory's theorem then assures us that standard conformal maps $f_N: \bar{B}(0, 1)^c \to \hat{\mathbb{C}}$ which represent A_N converge uniformly on compact subsets of $\bar{B}(0, 1)^c$ to a conformal map which represents A. We then show that this implies pointwise convergence of Green functions and weak convergence of the corresponding harmonic measures.

To this end, we recall Caratheodory's notion of convergence of domains.[2] If D_N, $N \geqslant 1$, is a sequence of simply connected open sets in $\hat{\mathbb{C}}$ containing a point z_0 in common, its *kernel* K is the connected component containing z_0 of the *strong limit inferior* of the D_N's, that is, of

$$\begin{aligned} s\text{-}\liminf_{N \to \infty} D_N &= \bigcup_{N=1}^{\infty} \operatorname{int} \bigcap_{j=N}^{\infty} D_j \\ &= \left\{ z: \exists \varepsilon > 0,\ N > 0 \text{ s.t. } B(z, \varepsilon) \subset \bigcap_{j=N}^{\infty} D_j \right\} \end{aligned} \tag{2.13}$$

When $z_0 = +\infty$ we can write $K = \operatorname{comp} s\text{-}\liminf_{N \to \infty} D_N$. The sequence D_N, $N \geqslant 1$, is said to converge if the kernel of any subsequence equals the kernel of the full sequence, in which case the sequence converges to its kernel. (Note that the kernel of a subsequence contains the kernel of the full sequence.)

The proof of the theorem rests on the next three lemmas.

Lemma 2.6. If $\lim_{N \to \infty} V_{A_N}(d\xi\, dx) = V(d\xi\, dx)$ and $A = \operatorname{comp}(spt\ \bar{V})^c$, then A_N converges to A in Caratheodory's sense.

Proof. Let $\tau(N)$, $N \geqslant 1$, be a subsequence of $N = 1, 2, 3, \ldots$, and let $A_{\tau(N)}$ be the corresponding subsequence of A_N. Let x_0 be a point of $s\text{-}\liminf_{N \to \infty} A_{\tau(N)}$, so that for some $\varepsilon > 0$ and $N_0 > 0$, $B(x_0, \varepsilon) \subset \bigcap_{N \geqslant N_0} A_{\tau(N)}$. If ϕ is any continuous function supported in $B(x_0, \varepsilon)$, then $\langle 1 \otimes \phi, V_{A_{\tau(N)}} \rangle = 0$ for $N \geqslant N_0$; hence $B(x_0, \varepsilon) \subset (spt\ \bar{V})^c$. It follows that

$$\bigcup_{\tau} s\text{-}\liminf_{N \to \infty} A_{\tau(N)} \subset (spt\ \bar{V})^c \tag{2.14}$$

On the other hand, let $x_0 \in (spt\ \bar{V})^c$. This set is open so for some $\varepsilon > 0$, $B(x_0, \varepsilon) \subset (spt\ \bar{V})^c$. Let us prove first that

$$\begin{gathered} \text{for all sufficiently large } N, \\ \text{either} \quad B(x_0, \varepsilon/4) \subset A_N \quad \text{or} \quad B(x_0, \varepsilon/4) \subset \bar{A}_N^c \end{gathered} \tag{2.15}$$

Note that for large N, ∂A_N cannot be contained in $B(x_0, \varepsilon)$, for this would contradict $B(x_0, \varepsilon) \subset (spt\ \bar{V})^c$. Thus, for large enough N, there are points $x_{N,1} \in \partial A_N \cap B(x_0, \varepsilon)^c$. Next, it is no loss of generality to assume

$$\liminf_{N \to \infty} \operatorname{dist}(\partial A_N, B(x_0, \varepsilon/4)) = 0$$

for otherwise (2.15) would hold. So there is a subsequence $\tau(N)$ and points $x_{\tau(N),2} \in \partial A_{\tau(N)} \cap B(x_0, \varepsilon/2)$. Let $J_{\tau(N)}$ be an arc of $\partial A_{\tau(N)}$ with endpoints $x_{\tau(N),1}$ and $x_{\tau(N),2}$. Then $J_{\tau(N)} \cap B(x_0, \varepsilon) \backslash B(x_0, \varepsilon/2)$ contains a subset of arc length at least $\varepsilon/2$, which contradicts the fact that $B(x_0, \varepsilon) \subset (spt\ \bar{V})^c$; hence, it is the case that

$$\liminf_{N \to \infty} \operatorname{dist}(\partial A_N, B(x_0, \varepsilon/4)) > 0$$

and so (2.15) does hold.

Now let us show that

$$\begin{aligned} &\text{either} \quad B(x_0, \varepsilon/4) \subset A_N \quad \text{for all large } N \\ &\text{or} \quad B(x_0, \varepsilon/4) \subset \bar{A}_N^c \quad \text{for all large } N \end{aligned} \tag{2.16}$$

Let ϕ be a continuous function supported in $B(x_0, \varepsilon/4)$ such that $\int \phi(x)\, dx = 1$. Define a vector field γ on $\mathbb{R}^2$ by $\gamma(x) = (1, \int_0^{x^2} \phi(x^1, u)\, du)$ and a function f on $S^1 \times \mathbb{R}^2$ by $f(\xi, x) = -\xi \cdot \gamma(x)$. Then f is bounded and continuous. Since A_N is a grid-square domain, we have by the divergence theorem for large N,

$$\begin{aligned} \langle f, V_{A_N} \rangle &= -\int n_{A_N}(x) \cdot \gamma(x)\, \sigma_{A_N}(dx) \\ &= \int_{A_N} \operatorname{div} \gamma(x)\, dx \\ &= \int_{A_N} \phi(x)\, dx \\ &= 1_{A_N}(x_0) \end{aligned} \tag{2.17}$$

due to (2.15). By hypothesis, $\langle f, V_{A_N} \rangle$, and hence $1_{A_N}(x_0)$, converges. But this implies the desired statement (2.16).

Finally, let

$$D = \bigcup_\tau s\text{-}\liminf_{N \to \infty} A_{\tau(N)}, \qquad E = s\text{-}\liminf_{N \to \infty} A_N, \qquad F = s\text{-}\liminf_{N \to \infty} \bar{A}_N^c$$

Lines (2.14) and (2.16) state that

$$D \subset (spt\ \bar{V})^c \subset E \cup F \tag{2.18}$$

and so

$$\text{comp}\ D \subset \text{comp}(spt\ \bar{V})^c \subset \text{comp}(E \cup F) \tag{2.19}$$

Now $K = \text{comp}\ E$ is the kernel of A_N, $N \geq 1$, and we have $K \subset \text{comp}\ D$ by definition. However, E and F are disjoint sets and $+\infty \in E$, in the sense of $\hat{\mathbb{C}}$, hence $\text{comp}(E \cup F) = \text{comp}\ E = K$. Therefore, by (2.19), $K = \text{comp}(spt\ \bar{V})^c$. ∎

Lemma 2.7. If $\lim_{N\to\infty} \sigma_{A_N}(dx) = \sigma(dx)$ and $A = \text{comp}(spt\ \sigma)^c$, then A_N converges to A in Caratheodory's sense.

Proof. Suppose $\tilde{A}$ is a limit point of A_N, $N \geq 1$, that is, suppose that for some subsequence $\tau(N)$, $A_{\tau(N)}$ converges to $\tilde{A}$ in Caratheodory's sense. By Proposition 2.3, $\lim_{N\to\infty} V_{A\tau'(N)} = V$, for some further subsequence τ', and $\bar{V} = \sigma$. Thus $\tilde{A} = A$ by Lemma 2.6 as applied to $A_{\tau'(N)}$. ∎

Lemma 2.8. If A_N converges to A, in Caratheodory's sense, and A contains a neighborhood of $+\infty$, then g_{A_N} converges pointwise to g_A on $\mathbb{R}^2$ and h_{A_N} converges weakly to h_A.

Proof. Recall that if f_N: $\bar{B}(0, 1)^c \to \mathbb{C}$, resp. f, are the Riemann maps representing A_N, resp. A, with a fixed normalization, say, $f(+\infty) = +\infty$ and $F'(0)$ is real, where $F(z) = f(1/z)$, then the corresponding Green functions can be written

$$g_{A_N}(x) = \log |\phi_N(x)|, \qquad g_A(x) = \log |\phi(x)| \tag{2.20}$$

where ϕ_N, ϕ are the inverse functions of f_N, f (ref. 10, p. 365).

Let us show that if $\limsup_{N\to\infty} g_{A_N}(z_0) > 0$, then $z_0 \in A$. Indeed, in this case, there is a subsequence $\tau(N)$ and a constant $0 < \kappa < 1$ such that

$$\kappa \leq g_{A\tau(N)}(z_0) \leq \kappa^{-1}, \qquad \text{hence} \quad e^{\kappa} \leq |\phi_{\tau(N)}(z_0)| \leq e^{\kappa^{-1}} \tag{2.21}$$

With the notation $\zeta_{\tau(N)} = \phi_{\tau(N)}(z_0)$, we see that $\zeta_{\tau(N)} \in B(0, e^{\kappa})^c \equiv F$. Since F is a compact set, in the sense of the Riemann sphere $\hat{\mathbb{C}}$, there is a further subsequence still denoted by τ such that $\zeta = \lim_{N\to\infty} \zeta_{\tau(N)}$ in $\hat{\mathbb{C}}$. By Caratheodory's theorem, $f_{\tau(N)}$ converges uniformly on F to f and so $z_0 = \lim_{N\to\infty} f_{\tau(N)}(\zeta_{\tau(N)}) = f(\zeta)$, which shows that $z_0 \in A$.

It follows that if $z_0 \notin A$, then $\lim_{N\to\infty} g_{A_N}(z_0) = 0$. But $g_A(\cdot)$ vanishes continuously on ∂A, and it is identically zero off $\bar{A}$. So pointwise convergence holds good in A^c.

On the other hand, suppose $z_0 \in A$. Because A is connected, there is a path γ in A with $+\infty$ and z_0 as endpoints. By compactness of γ in $\hat{\mathbb{C}}$ there are finitely many balls $B(y_i, \varepsilon_i)$, $0 \leqslant i \leqslant L$, with centers on γ such that if

$$T = \bigcup_{i=1}^{L} B(y_i, \varepsilon_i) \bigcup \bar{B}(y_0, \varepsilon_0)^c \tag{2.22}$$

Then $\gamma \subset T \subset A$ and hence $T \subset A_N$ for all large N. If $g_T(\cdot)$ is the Green function of the tube T with pole at $+\infty$, then

$$0 < g_T(z) \leqslant \min\{g_{A_N}(z), g_A(z)\}, \qquad z \in T \tag{2.23}$$

Thus, since $g_T(z_0) = \lambda > 0$, we have $\zeta_N = \phi_N(z_0) \in B(0, e^{\lambda/2})^c$, which is a compact set in $\hat{\mathbb{C}}$. It follows that $\lim_{N\to\infty} \zeta_N = \zeta = \phi(z_0)$, for otherwise, if some subsequence $\zeta_{\tau(N)}$ were to converge to $\bar{\zeta} \neq \zeta$ we would have, by uniform convergence in $B(0, e^{\lambda/2})^c$

$$z_0 = \lim_{N\to\infty} f_{\tau(N)}(\zeta_{\tau(N)}) = f(\bar{\zeta}) \neq z_0 \tag{2.24}$$

Thus we have

$$g_A(z_0) = \log|\zeta| = \lim_{N\to\infty} \log|\zeta_N| = \lim_{N\to\infty} g_{A_N}(z_0) \tag{2.25}$$

To check the convergence of h_{A_N}, let ϕ be a compactly supported smooth function. By the Riesz decomposition (ref. 6, pp. 11 and 52)

$$\phi(x) = \int \phi(y)\, h^x_{A_N}(dy) - \frac{1}{2\pi} \int \Delta\phi(y)\, g^x_{A_N}(y)\, dy \tag{2.26}$$

where the superscript denotes the location of the pole. In particular, for $x \to +\infty$ we obtain

$$\int \phi(y)\, h_{A_N}(dy) = \frac{1}{2\pi} \int \Delta\phi(y)\, g_{A_N}(y)\, dy \tag{2.27}$$

Now the vague convergence of h_{A_N} to h_A follows easily from Eq. (2.27) and the pointwise convergence of g_{A_N} to g_A. Since A_N, $N \geqslant 1$, and A contain neighborhoods of infinity, the measures h_{A_N} and h_A are all supported in some fixed ball, hence the vague convergence implies the apparently stronger weak convergence. ∎

Items (i) and (ii) of the theorem are now direct consequences of the previous three lemmas. To get item (iii) just note that by the proof of Lemma 2.8, $\lim_{N\to\infty} g_{A_N}(z) = g_A(z) > 0$ for every $z \in A$. So if ϕ is compactly supported in A, then because A_N converges to A in Caratheodory's sense, $spt\ \phi \subset A_N$ for all large N and

$$\langle \phi, 1_{C_n^c} \rangle = \langle \phi, 1_{A_N} \rangle = \langle \phi, H(g_{A_N}) \rangle = \int \phi(y)\, dy \tag{2.28}$$

Hence

$$\langle \phi, \mu \rangle = \lim_{N \to \infty} \langle \phi, 1_{C_N^c} \rangle = \int \phi(y)\, dy = \langle \phi, H(g_A) \rangle \quad \blacksquare$$

Corollary 2.9. Harmonic measure is a continuous function of the perimeter measure of connected sets: Let $\sigma \in \bar{\mathscr{V}}$ and define $h(\sigma)(dx) = h_A(dx)$, where $A = \text{comp}(spt\ \sigma)^c$. Let $\mathscr{M}_1$ be the space of probability measures on $\mathbb{R}^2$ under weak convergence. Then $h\colon \bar{\mathscr{V}} \to \mathscr{M}_1$ is a continuous function.

3. MARKOV CHAINS AND THE HELE-SHAW PROBLEM

Consider the sequence of Markov chains whose states are finite grid-squares

$$C_N = \bigcup_{y \in c_N} E_N(y), \qquad c_N \text{ a finite subset of } \frac{1}{N}\mathbb{Z}^2 \tag{3.1}$$

with transition probabilities

$$\text{Prob}(C_N \to C_N \cup E_N(y)) = h_N^*(C_N)(y) \tag{3.2}$$

where $h_N^*(C_N)$ is a probability measure on the nearest neighbors of C_N which approximates harmonic measure; for example, h_N^{RW}, the random walk hitting probabilities, or

$$h_N^{\text{BM}}(C_N)(y) = \int_{\partial C_N \cap E_N(y)} h(C_N^c)(dz), \qquad y \in C_N^c \cap \frac{1}{N}\mathbb{Z}^2 \tag{3.3}$$

and 0 otherwise, which is the probability that a Brownian particle, rather than a random walker, hits C_N near y. In this section we examine the possible convergence of these chains as $N \to \infty$.

For bounded continuous functions $\phi\colon \mathbb{R}^2 \to \mathbb{R}$, smooth functions $f\colon \mathbb{R} \to \mathbb{R}$, and finite clusters C_N, define the generator L_N by the formula

$$\begin{aligned} L_N f(\langle \phi, C_N \rangle) = \sum_{y \in (1/N)\mathbb{Z}^2} & [f(\langle \phi, C_N \cup E_N(y) \rangle) \\ & - f(\langle \phi, C_N \rangle)]\, N^2 h_N^*(C_N)(y) \end{aligned} \tag{3.4}$$

[here we use the notation $\langle \phi, C_N \rangle$ for $\langle \phi, 1_{C_N}(\cdot)\, dx \rangle = \int_{C_N} \phi(x)\, dx$] and set

$$\mathscr{E}_N f(\langle \phi, C_N \rangle) = \sum_{y \in (1/N)\mathbb{Z}^2} |f(\langle \phi, C_N \cup E_N(y) \rangle) - f(\langle \phi, C_N \rangle)|^2 N^2 h_N^*(C_N)(y) \tag{3.5}$$

Then there is a unique Markov chain, with measure values, such that

$$M_N^{f,\phi}(t) \equiv f(\langle \phi, X_N(t) \rangle) - f(\langle \phi, X_N(0) \rangle) - \int_0^t L_N f(\langle \phi, X_N(s^-) \rangle)\, dA_N(s) \tag{3.6}$$

is a mean-zero martingale, for all such ϕ and f. (Here $A_N(s) = [N^2 s]/N^2$.) A standard computation (see, e.g., ref. 8) shows that $M_N^{f,\phi}$ has quadratic variation process

$$[M_N^{f,\phi}](t) = \int_0^t \mathscr{E}_N f(\langle \phi, X_N(s^-) \rangle)\, dA_N(s) + O\left(\frac{1}{N^2}\right) \tag{3.7}$$

In what follows, we let $\mathscr{M}$ denote the space of σ-finite, Borel measures on $\mathbb{R}^2$ considered, for the sake of convenience, as Schwartz distributions; that is, convergence of measures is just convergence of integrals against Schwartz functions. This is, essentially, just vague convergence. Let $D([0, T], \mathscr{M})$ be the Skorokhod path space and $C([0, T], \mathscr{M})$ the subspace of continuous paths. Frequently we will use the notation

$$X_N(t) = 1_{C_N(t)}(x)\, dx \qquad \text{or} \qquad \langle \phi, X_N(t) \rangle = \langle \phi, C_N(t) \rangle \tag{3.8}$$

and

$$X_N^c(t) = 1_{C_N^c(t)}(x)\, dx \qquad \text{or} \qquad \langle \phi, X_N^c(t) \rangle = \int \phi(x)\, dx - \langle \phi, X_N(t) \rangle$$

If we want to display the initial state $X_N(0) = 1_{C_N}(x)\, dx$ we will write $X_N^{C_N}$ and $X_N^{C_n,c}$. The next proposition is included for the sake of emphasis.

Proposition 3.1. For any sequence of initial grid-square domains C_N, $N \geqslant 1$, such that $1_{C_N}(x)\, dx$ converges to $\mu \in \mathscr{M}$, both sequences $X_N^{C_N}$ and $X_N^{C_N,c}$ are tight on $D([0, T], \mathscr{M})$ and any limit point is concentrated on $C([0, T], \mathscr{M})$.

Proof. According to Mitoma's theorem,[16] it is enough just to check the tightness of the real-valued processes $\langle \phi, X_N^{C_N}(t) \rangle$ and $\langle \phi, X_N^{C_N,c}(t) \rangle$ for the Schwartz functions ϕ. But the tightness of these processes follows from the elementary facts that, by (3.5) and (3.6),

$$[M_N^{f,\phi}](t) \leqslant \text{const} \cdot T \, \|f'\|_\infty^2 \, \|\phi\|_\infty^2 \, N^{-2} \tag{3.9}$$

and that the considered processes have jumps no bigger than $1/N^2$ in size and, from an appeal to the standard martingale, sufficient conditions as exposed, for example, in ref. 8 and ref. 24, Chapter 1. ▮

Remark 3.2. It is possible, even likely, if Witten–Sander cluster have diameters of greater order of magnitude than the square root of their areas, that any such limit point on $C([0, T], \mathscr{M})$ is concentrated on constant trajectories, due to the circumstance that asymptotically in N, an arm of infinite perimeter but no area may grow immediately out to infinity. Thereafter, no area is added to the initial cluster in any finite part of the plane, as incoming Brownian particles or random walkers first meet the cluster at their starting point $+\infty$. These additions are undetected by integrating against Schwartz functions, which of course vanish at infinity.

Let $\bar{P}$ be any limit point guaranteed by the lemma. Naively, one would hope to prove that $\bar{P}$-almost surely, the canonical process $\mu_{\bullet}$ of $C([0, T], \mathscr{M})$ would solve some weak form of the Hele-Shaw problem. Indeed, the martingales $M_N^{f,\phi}$ converge to zero in probability thanks to (3.9), and of the three terms in the definition of $M_N^{f,\phi}$ in Eq. (3.6), the first two pass nicely to the limit, since for each fixed $0 \leqslant t \leqslant T$ the assignment $\mu_{\bullet} \to f(\langle \phi, \mu_t \rangle)$ is a $\bar{P}$-almost surely bounded, continuous function on $C([0, T], \mathscr{M})$. [Note that boundedness is due to the fact that all the measures $X_N^{C_N}(t)$ and $X_N^{C_N,c}(t)$ are dominated by two-dimensional Lebesgue measure.] However, the third term, which involves harmonic measure, presents a problem due to the fact that the assignment

$$\mu \to h(spt\,\mu)(dy) \tag{3.10}$$

is not a continuous function of $\mu \in \mathscr{M}$ under vague convergence. Thus, according to Theorem 2.5, if we wish to pick up the Hele-Shaw problem in the limit, we are obliged to track not only cluster area, but also cluster perimeter.

For technical reasons we also require a degree of approximation of $h(C_N)(dx)$ by $h_N^*(C_N)(dx)$ which is uniform in the set variable and the following simple lemma provides us with more than enough in case $h_N^* = h_N^{\mathrm{BM}}$. For the corresponding result in the more interesting case $h_N^* = h_N^{\mathrm{RW}}$, consult the Appendix.

Lemma 3.3. Let ϕ be a C^1 function with bounded gradient. For any finite grid-square C_N,

$$|\langle \phi, h_N^{\mathrm{BM}}(C_N^c)\rangle - \langle \phi, h(C_N^c)\rangle| \leqslant \frac{2\,\|\nabla\phi\|_\infty}{N}$$

Proof. Notice that if $y \in (1/N)\,\mathbb{Z}^2 \cap C_N^c$, then

$$\int_{\partial C_N^c \cap E_N(y)} \phi(z)\, h(C_N^c)\, dz = \phi(y)\, h_N^{\mathrm{BM}}(C_N)(y)$$
$$+ \int_{\partial C_N^c \cap E_N(y)} [\phi(z) - \phi(y)]\, h(C_N^c)(dz) \qquad (3.11)$$

Thus

$$\left| \sum_{y \in C_N^c \cap (1/N)\,\mathbb{Z}^2} \phi(y)\, h_N^{\mathrm{BM}}(C_N)(y) - \int \phi(z)\, h(C_N^c)(dz) \right|$$
$$\leqslant \sum_{y \in C_N^c \cap (1/N)\,\mathbb{Z}^2} \int_{\partial C_N^c \cap E_N(y)} |\phi(z) - \phi(y)|\, h(C_N^c)(dz)$$
$$\leqslant \frac{2\,\|\nabla\phi\|_\infty}{N} \quad \blacksquare \qquad (3.12)$$

If C_N is a connected grid-square domain, and if we write

$$C_N^c = A_N \cup B_N \qquad (3.13)$$

as in the previous section, then we determine a pair of measures

$$(1_{C_N^c}(x)\, dx,\, \sigma_{A_N}(d\xi)) \in \mathscr{M} \times \bar{\mathscr{V}} \qquad (3.14)$$

so that the chains $X_N^{C_N, c}(\cdot)$, $N \geqslant 1$, determine via (3.14) processes with state space $\mathscr{M} \times \bar{\mathscr{V}}$ whose probability laws Q_N are defined on $D([0, T], \mathscr{M} \times \bar{\mathscr{V}})$. The next result concerns the consequence of assuming Q_N, $N \geqslant 1$, has a limit point on $C([0, T], \mathscr{M} \times \bar{\mathscr{V}})$. We remark that by Proposition 3.1 the marginals of Q_N on $D([0, T], \mathscr{M})$ already enjoy this tightness property.

Theorem 3.4. Let (μ_t, σ_t) be the canonical process on $D([0, T], \mathscr{M} \times \bar{\mathscr{V}})$ and define $A_t = \mathrm{comp}(spt\, \sigma_t)^c$, the unbounded, connected component of the complement of the closed support of σ_t. Define also $u(x, t) = (1/2\pi)\, g_{A_t}(x)$, the Green function of A_t with pole at $+\infty$, and

$$\beta_t(dx) = \mu_t|_{\{u(\cdot, t) = 0\}}(dx)$$

the measure of *bubbles*. If Q is any limit point of Q_N on $C([0, T], \mathscr{M} \times \bar{\mathscr{V}})$, then Q-almost surely, for $0 \leqslant t \leqslant T$, and Schwartz functions ϕ,

$$\langle \phi, H(u(\cdot, t)) \rangle - \langle \phi, H(u(\cdot, 0)) \rangle + \langle \phi, \beta_t \rangle = -\int_0^t \langle \Delta\phi, u(\cdot, s) \rangle \, ds$$

Proof. The theorem follows directly from an adaptation of the proof of Theorem 3.3.1 of ref. 8, concerning convergence of sequences of semimartingales, as applied to $\langle \phi, X_N^{C_N, c}(\cdot) \rangle$. Since

$$\langle \phi, X_N^{C_N, c}(t) \rangle - \langle \phi, X_N^{C_N, c}(0) \rangle = \langle \phi, X_N^{C_N}(0) \rangle - \langle \phi, X_N^{C_N}(t) \rangle$$

we have from (3.4) and (3.6),

$$\begin{aligned} R_N^{\phi}(t) \equiv {} & \langle \phi, X_N^{C_N, c}(t) \rangle - \langle \phi, X_N^{C_N, c}(0) \rangle \\ & + \int_0^t \sum_y \langle \phi, E_N(y) \rangle \, N^2 h_N^{\mathrm{RW}}(C_N(s^-))(y) \, dA_N(s) \end{aligned} \tag{3.15}$$

is a mean-zero martingale and by (3.9) its quadratic variation process satisfies the bound

$$[R_N^{\phi}](t) \leqslant \frac{KT \, \|\phi\|_{\infty}^2}{N^2} \tag{3.16}$$

By Doob's inequality we have

$$E[\sup_{0 \leqslant t \leqslant T} |R_N^{\phi}(t)|^2] \leqslant \frac{4KT \, \|\phi\|_{\infty}^2}{N^2} \tag{3.17}$$

and so the convergence of $R_N^{\phi}(\cdot)$ to the zero process in probability.

By Theorem 2.5 and Corollary 2.9 the map $\Phi\colon \mathscr{M} \times \bar{\mathscr{V}} \to \mathbb{R}^2$ defined by $\Phi(\mu, \sigma) = (\langle \phi, \mu \rangle, \langle \phi, h(\sigma) \rangle)$, where $h(\sigma)(dx) = h_A(dx)$ and $A = \mathrm{comp}(spt\, \sigma)^c$, is continuous. It follows that for $0 \leqslant s \leqslant t \leqslant T$,

$$\Phi_{s,t}(\mu_\cdot, \sigma_\cdot) \equiv \langle \phi, \mu_t \rangle - \langle \phi, \mu_s \rangle + \int_s^t \langle \phi, h(\sigma_{s^-}) \rangle \, ds \tag{3.18}$$

is a real-valued function on $D([0, T], \mathscr{M} \times \bar{\mathscr{V}})$ which is continuous on $C([0, T], \mathscr{M} \times \bar{\mathscr{V}})$ and is Q-almost surely bounded there. Just below we will want to be able to write $R_N^{\phi}(t)$ as a function of the canonical process. To do this, set

$$h_N^*(\sigma) = h_N^*(A), \qquad \text{where} \quad A = \mathrm{comp}(spt\, \sigma)^c$$

if σ is the perimeter measure of a grid-square domain of mesh $1/N$ and set it to zero otherwise. Introduce

$$\Phi_{N,s,t}(\mu_{\cdot}, \sigma_{\cdot}) = \langle \phi, \mu_t \rangle - \langle \phi, \mu_s \rangle + \int_s^t \sum_y \langle \phi, E_N(y) \rangle N^2 h_N^*(\sigma_{s^-})(y)\, dA_N(s) \tag{3.19}$$

Then clearly $R_N^\phi(t) - R_N^\phi(s)$ has the same distribution as $\Phi_{N,s,t}$ under Q_N.

Now let ψ_s be a bounded, continuous, $\mathscr{F}_s$ measurable function. Then

$$\begin{aligned} 0 &= E(\psi_s[R_N^\phi(t) - R_N^\phi(s)]) \\ &= E^{Q_N}(\psi_s \Phi_{N,s,t}) \end{aligned} \tag{3.20}$$

since R_N^ϕ is a martingale. Thus,

$$\begin{aligned} E^{Q_N}(\psi_s \Phi_{s,t}) &= E^{Q_N}(\psi_s[\Phi_{s,t} - \Phi_{N,s,t}]) \\ &= E^{Q_N}\left(\psi_s\left[\int_s^t \langle \phi, h(\sigma_{s^-}) \rangle\, ds \right.\right. \\ &\quad \left.\left. - \int_s^t \sum_y \langle \phi, E_N(y) \rangle N^2 h_N^*(\sigma_{s^-})\, dA_N(s)\right]\right) \\ &\leqslant E^{Q_N}\left(\psi_s\left[\int_s^t \langle \phi, h(\sigma_{s^-}) \rangle\, ds - \int_s^t \langle \phi, h_N^*(\sigma_{s^-}) \rangle\, ds\right]\right) \\ &\quad + E^{Q_N}\left(\psi_s\left[\int_s^t \langle \phi, h_N^*(\sigma_{s^-}) \rangle\, ds \right.\right. \\ &\quad \left.\left. - \int_s^t \langle \phi, h_N^*(\sigma_{s^-}) \rangle\, dA_N(s)\right]\right) \\ &\quad + E^{Q_N}\left(\psi_s \int_s^t \left[\langle \phi, h_N^*(\sigma_{s^-}) \rangle \right.\right. \\ &\quad \left.\left. - \sum_y \langle \phi, E_N(y) \rangle N^2 h_N^*(\sigma_{s^-})(y)\right] dA_N(s)\right) \end{aligned} \tag{3.21}$$

Thus, by Lemma 3.3, in the case $h_N^* = h_N^{\mathrm{BM}}$ (or by the Appendix if $h_N^* = h_N^{\mathrm{RW}}$),

$$\begin{aligned} |E^{Q_N}(\psi_s \Phi_{s,t})| &\leqslant \frac{2K \|\phi\|_\infty (t-s)}{N} \\ &\quad + K(t-s)\, \|\phi\|_\infty \{t - s - [A_N(t) - A_N(s)]\} \\ &\quad + \frac{K \|\nabla\phi\|_\infty\, |A_N(t) - A_N(s)|}{N} \end{aligned} \tag{3.22}$$

hence

$$\lim_{N \to \infty} E^{Q_N} \psi_s \Phi_{s,t} = 0 \tag{3.23}$$

On the other hand, since $\psi_s \Phi_{s,t}$ is a Q-almost surely bounded, continuous function

$$0 = \lim_{N \to \infty} E^{Q_N} \psi_s \Phi_{s,t} = E^{Q} \psi_s \Phi_{s,t} \tag{3.24}$$

This, together with (3.17), shows that under Q, $\Phi_{0,t}$ is just the zero process, so that Q-almost surely

$$\langle \phi, \mu_t \rangle - \langle \phi, \mu_0 \rangle = -\int_0^t \langle \phi, h(\sigma_s) \rangle \, ds \tag{3.25}$$

However, the left-hand side can be rewritten, according to Theorem 2.6, as

$$\langle \phi, H(u(\cdot, t)) \rangle - \langle \phi, H(u(\cdot, 0)) \rangle + \langle \phi, \beta_t \rangle \tag{3.26}$$

and the integrand on the right-hand side, according to the Riesz decomposition (2.26)–(2.27), as $\langle \Delta\phi, u(\cdot, s) \rangle$ and this finishes the proof. ∎

Remark 3.5. We could just as well have considered the space-time process $(t, X^{C_N, c}(t))$ with generator $\partial/\partial t + L_N$. Then for time-dependent test functions ϕ and assuming $\beta_s = 0$, $s \leqslant t$, one would get as the limiting equation

$$\langle \phi(\cdot, t), H(u(\cdot, t)) \rangle - \langle \phi(\cdot, 0), H(u(\cdot, 0)) \rangle - \int_0^t \left\langle \frac{\partial}{\partial s} \phi(\cdot, s), H(u(\cdot, s)) \right\rangle ds$$

$$= -\int_0^t \langle \Delta\phi(\cdot, s), u(\cdot, s) \rangle \, ds$$

which is a weak form of the backward nonlinear heat equation:

$$\frac{\partial}{\partial t} H(u(x, t)) = -\Delta u(x, t)$$

This equation was also obtained from the formulation (1.3a), (1.3b) of the Hele-Shaw problem, where $\lambda = 1$ and $\sigma = 0$. ∎

Let $\chi = \inf\{t > 0 : \beta_t > 0\}$ be the *condensation* time, namely the first instant that the support of μ_t contains a set of *bubbles* of positive μ_t-measure. Our next result concerns the consequences which follow from assuming $\chi > 0$. The ill-posedness of this problem has also been discussed in ref. 5.

Theorem 3.6. Let A_t, $0 \leqslant t \leqslant T$, be a one parameter family of unbounded, decreasing, simply connected neighborhoods of $+\infty$ and let $u(x, t) = (1/2\pi)\, g_{A_t}(x)$ be the corresponding Green functions with pole at $+\infty$. Suppose that for all Schwarz functions ϕ,

$$\langle \phi, H(u(\cdot, t))\rangle - \langle \phi, H(u(\cdot, 0))\rangle = -\int_0^t \langle \Delta\phi, u(\cdot, s)\rangle\, ds$$

Finally suppose ∂A_0 is a Jordan curve in the plane with zero Lebesgue measure. Then either (i) ∂A_0 contains an analytic arc or (ii) $\partial A_0 \subset \partial A_t$ for all $0 \leqslant t \leqslant T$ and $\int_0^T h_{A_s}(\partial A_0)\, ds = 0$.

Proof. First suppose $\partial A_0 \backslash \partial A_t \neq \varnothing$ for some $0 < t \leqslant T$. Since $\partial A_0 \backslash \partial A_t$ is a nonempty open subset of ∂A_0, there exists an open subarc $J \subset \partial A_0$ with $J \cap \partial A_t = \varnothing$. If $y \in J$, then for some $\varepsilon > 0$, $B(y, \varepsilon) \cap A_0 \subset \bar{A}_t^c$, for otherwise, since $A_t \subset A_0$, there exists a sequence of radii $\varepsilon_n \downarrow 0$ such that $(B(y, \varepsilon_n) \cap A_0) \cap \bar{A}_t \neq \varnothing$, hence a sequence of points $\xi_n \in \partial A_t$ such that $\lim_{n\to\infty} \xi_n = y$. Because ∂A_t is closed, this implies $y \in \partial A_t$, in contradiction to the fact that $y \in J$.

Now consider the function

$$v_t(x) = \int_0^t u(x, s)\, ds \tag{3.27}$$

Since each $u(\cdot, t)$ is continuous on $\mathbb{R}^2$ (ref. 10, p. 365), $v_t(\cdot)$ is also continuous and we have

$$\langle \Delta\phi, v_t\rangle = \langle \phi, H(u(\cdot, 0))\rangle - \langle \phi, H(u(\cdot, t))\rangle \tag{3.28}$$

If ϕ is supported in $B(y, \varepsilon)$, then since this ball is disjoint from A_t,

$$\langle \phi, H(u(\cdot, t))\rangle = \int_{A_t} \phi(x)\, dx = 0 \tag{3.29}$$

$$\langle \phi, H(u(\cdot, 0))\rangle = \int_{A_0} \phi(x)\, dx \tag{3.30}$$

Thus v_t is a continuous function satisfying the equation

$$\frac{1}{2} \Delta v_t(x) = 1_{B(y,\varepsilon) \cap A_0}(x) \qquad \text{in} \quad B(y, \varepsilon) \tag{3.31}$$

in weak form, and so it is a C^1 function in $B(y, \varepsilon)$ (ref. 6, p. 8). In particular, v_t solves the equation

$$\Delta v_t = 1 \qquad \text{in} \quad B(y, \varepsilon) \cap A_0 \tag{3.32}$$

Moreover, since $A_s \subset A_0$, we have $u(x, s) \geqslant 0$ and $u(x, s) = 0$ on ∂A_0. This implies $v_t(x) \geqslant 0$ and $v_t(x) = 0$ on ∂A_0. So since v_t is a C^1-function in $B(y, \varepsilon)$ which achieves its minimum on $\partial A_0 \cap B(y, \varepsilon)$, both

$$v_t = 0 \qquad \text{and} \qquad \nabla v_t = 0 \qquad \text{on} \quad \partial A_0 \cap B(y, \varepsilon) \tag{3.33}$$

Therefore, according to Theorem 2.1, p. 139, of ref. 7, (3.32) and (3.33) imply that $\partial A_0 \cap B(y, \varepsilon)$ admits of an analytic parametrization.

Now let us assume that $\partial A_0 \backslash \partial A_t = \varnothing$ for all $0 < t \leqslant T$, which means $\partial A_0 \subset \partial A_t$. Let ϕ_n be a sequence of smooth, compactly supported functions such that

$$0 \leqslant \phi_n \leqslant 1, \qquad \phi_n \equiv 1 \qquad \text{on} \quad \partial A_0, \qquad \lim_{n \to \infty} \phi_n = 0 \quad \text{off} \quad \partial A_0 \tag{3.34}$$

(These can be constructed using the regularized distance; see Theorem 2, p. 171, of ref. 23.) Using again the fact that $\langle \Delta \phi_n, u(\cdot, s) \rangle = \langle \phi_n, h_{A_s} \rangle$, we have

$$\begin{aligned}
\int_0^T h_{A_s}(\partial A_0)\, ds &\leqslant \int_0^T \langle \phi_n, h_{A_s} \rangle\, ds \\
&= \int_0^T \langle \Delta \phi_n, u(\cdot, s) \rangle\, ds \\
&= \langle \phi_n, H(u(\cdot, T)) \rangle - \langle \phi_n, H(u(\cdot, 0)) \rangle \\
&\leqslant \int \phi_n(x)\, dx
\end{aligned} \tag{3.35}$$

Letting $n \to \infty$ and using the fact that ∂A_0 has two-dimensional Lebesgue measure zero, it develops that $\int_0^T h(A_s)(\partial A_0)\, ds = 0$, which ends the proof. ▌

We are grateful to a referee, who pointed out to us the following important improvement to the previous theorem.

Proposition 3.7. Let A_t, $0 \leqslant t \leqslant T$, be as in Theorem 3.6 and suppose that ∂A_0 has positive one-dimensional Hausdorff measure. If $A_t = \text{comp}(spt\, \sigma_t)^c$ for some $\sigma_t \in \mathscr{V}$, then item (ii) of that theorem cannot hold.

Proof. Recall that a domain $B < \mathbb{R}^2$ has a rectifiable boundary if and only if

$$\sup\left\{\sum_{i=1}^{n} |\xi_i - \xi_{i-1}| \,\middle|\, n \geqslant 1; \xi_i \in \partial B, i = 0,..., n; \xi_n = \xi_0\right\} < \infty$$

Each collection $\{\xi_0,..., \xi_n\}$ determines a piecewise C^1 domain $B(\xi_1,..., \xi_n)$ and a corresponding perimeter measure $\bar{V}_{B(\xi_1,..., \xi_n)}$. Since for any $(\alpha, \beta) \in \mathbb{R}^2$,

$$\frac{|\alpha| + |\beta|}{\sqrt{2}} \leqslant |(\alpha, \beta)| \leqslant |\alpha| + |\beta|$$

it follows from the proof of Proposition 2.2 that

$$\frac{1}{\sqrt{2}} \bar{V}_{B(\xi_1,..., \xi_n)}(\mathbb{R}^2) \leqslant \sum_{I=1}^{n} |\xi_i - \xi_{i-1}| \leqslant \bar{V}_{B(\xi_1,..., \xi_n)}(\mathbb{R}^2)$$

Hence ∂B is rectifiable if and only if

$$\sup\{\bar{V}_{B(\xi_1,..., \xi_n)}(\mathbb{R}^2) \mid n \geqslant 1; \xi_i \in \partial B, i = 1,..., n; \xi_0 = \xi_n\} < \infty$$

Each $\bar{V}_{B(\xi_1,..., \xi_n)}$ is the limit perimeter measure of approximating grid-square domains; hence the supremum above can be replaced by the supremum over all grid-square domains which approximate each $B(\xi_1,..., \xi_n)$ as in Proposition 2.2. Therefore, if $B = \text{comp}(spt\,\sigma)^c$, where $\sigma \in \mathscr{V}$, then since $\sigma(\mathbb{R}^2) < \infty$ and σ is the limit along a subsequence of such grid-square domains, ∂B is rectifiable.

Suppose ∂B is rectifiable and let $f: D \to B$ be the corresponding Riemann map from the unit disk D. According to Theorem 10.12 of Pommerenke,[17] if $A \subset \partial D$, then the Lebesgue measure of A is zero if and only if the one-dimensional Hausdorff measure of $f(A) \subset \partial B$ is zero. On the other hand, since h_D and Lebesgue measure on ∂D are mutually absolutely continuous,

$$h_B(f(A)) = h_D(A) = 0$$

if and only if the Lebesgue measure of A is zero. Thus, a measurable subset of ∂B has h_B measure zero if and only if it has one-dimensional Hausdorff measure zero. Therefore, if $B = A_t$, then $h_{A_t}(\partial A_0) = 0$ implies ∂A_0 has one-dimensional Hausdorff measure zero, which is a contradiction. ∎

Finally, we summarize the possible outcomes of scaling a sequence of Witten–Sander type models.

Theorem 3.8. Let C_N, $N \geqslant 1$, be a sequence of grid-square domains of mesh $1/N$, let $X_N^{C_N,c}(t, dx) = 1_{C_N^c(t)}(x)\,dx$ be the Markov chains defined at (3.4)–(3.6), and let $Q_N^{C_N}$ be the probability laws of $(1_{C_N^c(t)}(x)\,dx, \sigma_{A_N(t)}(d\xi))$ on the space $D([0, T], \mathscr{M} \times \bar{\mathscr{V}})$. Let $\sigma_{C_N}(d\xi)$ converge to $\bar{V}_C(d\xi)$ for some piecewise C^1-Jordan domain C such that ∂C contains no analytic arc. If the sequence $Q_N^{C_N}$, $N \geqslant 1$, has a limit point Q on $C([0, T], \mathscr{M} \times \bar{\mathscr{V}})$, then Q-almost surely, $\chi = 0$.

Proof. This is a direct consequence of Proposition 2.2, Theorem 3.4, Theorem 3.6, and Proposition 3.7.

4. DISCUSSION AND FURTHER QUESTIONS

It is well understood that the long arms of Witten–Sander clusters, as simulated on the computer, are due to the fact that the tips of projections are favored hitting sites for incoming random walkers so that existing projections are likely to grow even longer at the expense of other parts of the boundary, which become effectively screened. However, a rigorous statement to this effect is hard to come by because the very feature of interest, namely the intricately growing network of arms and projections, makes quantitative estimates of random walk hitting probabilities hard to prove. This motivates a study of the effect of regularization of the cluster boundary between successive additions of particles. Notice that while adding N^2 particles to a cluster in $(1/N)\,\mathbb{Z}^2$ adds one unit of area, it could add up to $O(N)$ units of perimeter if a fixed fraction of new particles attach themselves at boundary sites having only one nearest neighbor in the cluster. A well-chosen mechanism for rearrangement of occupied sites might tend to detach particles having just one nearest neighbor in the cluster and reattach them at a site with two or more such neighbors. Then cluster area would be conserved but cluster perimeter decreased on average. Presumably such regularization would lead in the continuum limit to a moving boundary problem which includes a term interpretable as surface tension.

A Monte Carlo algorithm for simulation of Hele-Shaw problems with surface tension has been proposed by Kadanoff[9] and Szep *et al.*[25] and simulations have been carried out, with delightful results, by Liang.[15] Convergence of this algorithm, or perhaps some simplification of it, is an interesting problem which it may be possible to discuss within the framework outlined in the previous sections. We remark only that this problem bears close resemblance to the derivation of reaction-diffusion equations as hydrodynamic limits of interacting particle systems (e.g., refs. 3 and 14). Here rearrangement of boundary sites is analogous to diffusion and addition of new sites by random walkers from infinity is analogous to reaction. Indeed this has essentially been done in ref. 4 for a highly simplified model of surface tension introduced by Plischke *et al.*[19] In this model, movement of the boundary is determined by the development of a periodic exclusion process on $\mathbb{Z}$. Passage to the continuum limit corresponds exactly to the hydrodynamic limit.

A different approach to regularization is to alter the rule for the addition of new points to the cluster. This approach is taken by Kesten.[13]

APPENDIX

The results of this section are wholly based on ideas provided to the author by H. Kesten. It is a pleasure to thank him for his permission to present the results here.

Let D be either a domain in the plane $\mathbb{R}^2$ or the complement of one. We consider the problem of approximating the classical harmonic measure of D, that is, the hitting distribution of Brownian motion, by the hitting distribution, on a related subset $D_N \subset (1/N)\,\mathbb{Z}^2$, of a simple random walk.

Let $D_N = D \cap (1/N)\,\mathbb{Z}^2$. If $x \in (1/N)\,\mathbb{Z}^2$, let $e(x)$ be the union of the line segments joining x to its four nearest neighbors. Define $\partial_N D = \{x \in D_N \mid e(x) \cap D^c \neq \varnothing\}$ and set $D_N^0 = D_N \backslash \partial_N D$. Thus $D_N \subset D$ and points of $\partial_N D$ are at a distance less than $1/N$ from the boundary of D.

If ϕ: $\mathbb{R}^2 \to \mathbb{R}$ is a smooth, bounded function, let u be the solution of the Dirichlet problem:

$$\begin{cases} \Delta u(x) = 0 \quad \text{in} \quad D \\ u\,|_{\partial D}(\xi) = \phi(\xi) \end{cases}$$

and let u_N be the solution of the discrete Dirichlet problem

$$\begin{cases} \Delta_N u_N(x) = 0 \quad \text{in} \quad D_N^0 \\ u_N\,|_{\partial_N D}(\xi) = \phi(\xi) \end{cases}$$

where Δ_N is the discrete or five-point Laplacian. Of course, u and u_N are averages of ϕ with respect to the hitting distributions of Brownian motion and simple random walk, respectively. Thus, estimates on $\sup_{x \in D_N} |u(x) - u_N(x)|$ gauge the degree of approximation of the former by the latter.

Theorem A. If u_N and u are solutions to the discrete and classical Dirichlet problems on D_N and D, respectively, then

$$\sup_{x \in D_N} |u(x) - u_N(x)| \leqslant C(\|\phi\|_\infty + \|\nabla\phi\|_\infty)\, N^{-1/11}$$

for some universal constant C independent of D, N, and ϕ.

The following lemmas prepare us for the proof of the theorem. We use the notation $d(x, y)$ for the number of lattice steps in any shortest path in $(1/N)\,\mathbb{Z}^2$ whose endpoints are x and y. If $x \in D_N^0$, let $\delta(x)$ be the number of steps in any shortest path in $(1/N)\,\mathbb{Z}^2$ having one endpoint in $\partial_N D$ and the other at x. In particular,

$$\frac{d(x, y)}{\sqrt{2}\,N} \leqslant |x - y| \leqslant \frac{d(x, y)}{N}$$

and

$$\frac{\delta(x)}{\sqrt{2}\,N} \leqslant \operatorname{dist}(x, \partial D) \leqslant \frac{\delta(x) + 1}{N}$$

Lemma A.1. Let u be the harmonic function with boundary values ϕ. Let $x \in D$ and $y \in \partial D$. Then

$$|u(x) - \phi(y)| \leqslant C(\|\phi\|_\infty + \|\nabla\phi\|_\infty)\,|x - y|^{1/3}$$

Proof. By a translation of coordinates, we may assume that the boundary point y is at the origin. Let $\lambda = |x - y| = |x|$ and define $B = B(0, \lambda^\beta)$, $0 < \beta < 1$; $\tilde{B} = B(0, 1)$; and $\tilde{D} = \{\lambda^{-\beta} z \mid z \in D\}$. Note that we need only consider the case $|x| < 1$, since, by the maximum principle, the inequality is vacuous otherwise. In case $|x| < 1$, note that $x \in B$ since $\lambda < 1$. For any domain C, let $\tau_C = \inf\{t > 0 \mid X_t \notin C\}$, where X is Brownian motion.

By a classical estimate of harmonic measure (e.g., ref. 17, Exercise 10, p. 352)

$$\begin{aligned} P_x\{\tau_B \leqslant \tau_D\} &= P_{\lambda^{-\beta}x}\{\tau_{\tilde{B}} \leqslant \tau_{\tilde{D}}\} \\ &\leqslant 1 - \frac{2}{\pi} \arcsin\left(\frac{1 - \lambda^{-\beta}\,|x|}{1 + \lambda^{-\beta}\,|x|}\right) \\ &\leqslant C\lambda^{(1-\beta)/2} \end{aligned}$$

Now it follows that for $\tau = \tau_B \wedge \tau_D$,

$$\begin{aligned} |u(x) - \phi(y)| &\leqslant E_\lambda\{|\phi(X_\tau) - \phi(y)|;\, \tau_B \leqslant \tau_D\} + E_x\{|\phi(X_\tau) - \phi(y)|;\, \tau_B > \tau_D\} \\ &\leqslant 2\,\|\phi\|_\infty P_x\{\tau_B \leqslant \tau_D\} + \sup_{|z - x| \leqslant \lambda^\beta} |\phi(z) - \phi(y)| \\ &\leqslant C(\|\phi\|_\infty \lambda^{(1-\beta)/2} + \|\nabla\phi\|_\infty \lambda^\beta) \end{aligned}$$

To conclude the proof, choose $\beta = 1/3$. ∎

Lemma A.2. Let $x \in D_N$ with $\delta(x) = k \geqslant 1$. If u is the harmonic function in D with boundary values ϕ, then for some universal constant C, the following hold:

(i) $|\Delta_N u(x)| \leqslant C(\|\phi\|_\infty + \|\nabla\phi\|_\infty)[(k+2)/N]^{1/3}$.

(ii) If $k \geqslant 2$, then $|\Delta_N u(x)| \leqslant C\, \|\phi\|_\infty / k^3$.

Proof. Coming first to the proof of (ii), note that

$$\frac{k}{\sqrt{2}\,N} \leqslant \operatorname{dist}(x, \partial_N D) \leqslant \operatorname{dist}(x, \partial D) \leqslant \frac{k+1}{N}$$

so that if $\sqrt{2} < s < 2$, then u is harmonic in the ball of radius k/Ns about x. Now define $\tilde{u}(y) = u[x + (k/Ns)\, y]$. Then $\tilde{u}$ is harmonic in the unit disk and so can be represented as the Poisson integral of its restriction $\bar{u}$ to the unit circle. Since the third derivatives of the Poisson kernel are uniformly bounded on the disk of radius $s/2$, we have by Taylor's formula

$$\begin{aligned}\Delta_N u(x) &= \frac{1}{4}\left[\tilde{u}\left(\frac{s}{k}e_1\right) + \tilde{u}\left(\frac{-s}{k}e_1\right) + \tilde{u}\left(\frac{s}{k}e_2\right)\right.\\ &\qquad \left. + \tilde{u}\left(\frac{-s}{k}e_2\right) - 4\tilde{u}(0)\right]\\ &\leqslant C\,\|\bar{u}\|_\infty \left(\frac{s}{k}\right)^3 \leqslant \frac{C'\,\|\phi\|_\infty}{k^3}\end{aligned}$$

As for the proof of (i), let y be a nearest point of ∂D to x. Since $\operatorname{dist}(x, \partial D) \leqslant (1+k)/N$, we have $|x - y| \leqslant (k+1)/N$ and $|x \pm e_0/N - y| \leqslant (k+2)/N$, $i = 1, 2$. By Lemma A.1, both $|u(x) - \phi(y)|$ and $|u(x \pm e_i) - \phi(y)|$ are bounded above by

$$C(\|\phi\|_\infty + \|\nabla\phi\|_\infty)\left(\frac{k+2}{N}\right)^{1/3}$$

By adding and subtracting $\phi(y)$ appropriately in the sum defining $\Delta_N u(x)$, we find $|\Delta_N u(x)|$ is bounded above by a term of the same form but with a slightly larger constant. ∎

Lemma A.3. For each $\varepsilon > 0$ there exists a constant C, depending only on ε, such that for each $k \geqslant 0$

$$\sup_{x \in D_N} \sum_{\substack{\delta(y) \leqslant k \\ y \in D_N}} g_{D_N}(x, y) \leqslant C(1+k)^{2+\varepsilon} \log(1+k)$$

where g_{D_N} is the Green function for Δ_N in D_N^0.

Proof. For each $x \in D_N$ with $\delta(x) \leqslant k$, choose $w = w(x)$ in $\partial_N D$ with $d(x, w) \leqslant k$ and define

$$B(x) = \left\{ z \in \frac{1}{N} \mathbb{Z}^2 \,\middle|\, |x - w(x)| \leqslant \frac{1}{N} (1+k)^{1+\varepsilon/2} \right\}$$

and set $\tau = \min\{ j \geqslant 0 \mid S_j \notin B(S(0))\}$, where S is simple random walk in $(1/N)\,\mathbb{Z}^2$.

First we show there is a number $\alpha = \alpha(\varepsilon) > 0$ such that if $x \in D_N$ and $\delta(x) \leqslant k$, then $P_x\{\tau \leqslant \tau_{D_N}\} \leqslant 1 - \alpha$, where $\tau_{D_N} = \min\{ j \geqslant 0\colon S_j \in \partial_N D\}$. Notice that by a translation of coordinates we can assume that $w(x)$ is at the origin. Since $\eta \equiv \tau \wedge \tau_{D_N} \leqslant \tau_0$, the first hitting time of the origin, it follows from Lemma 1 of ref. 12 that $n \to a_N(S_{n \wedge \eta})$ is a nonnegative martingale, where $a_N(x) = a(Nx)$ and $a(\cdot)$ is the potential kernel of simple random walk on $\mathbb{Z}^2$. Referring again to Lemma 1 of ref. 12, one sees that

$$a_N(x) = \frac{2}{\pi} \log |Nx| + C_0 + O(|Nx|^{-2})$$

for some constant C_0. Arguing exactly as in Lemma 3 of ref. 12, we find

$$\begin{aligned} a_N(x) &= E_x\{a_N(S_\eta); \tau \leqslant \tau_{D_N}\} + E_x\{a_N(S_\eta); \tau > \tau_{D_N}\} \\ &\geqslant P_x\{\tau \leqslant \tau_{D_N}\}\, E_x\{a_N(S_\eta) \mid \tau \leqslant \tau_{D_N}\} \end{aligned}$$

Since $|S_\eta| \geqslant |1+k|^{1+\varepsilon/2}/N$ on $\{\tau \leqslant \tau_{D_N}\}$, it follows that

$$\begin{aligned} P_x\{\tau \leqslant \tau_{D_N}\} &\leqslant a_N(x)[E_x\{a_N(S_\eta) \mid \tau \leqslant \tau_{D_N}\}]^{-1} \\ &= \frac{(2/\pi) \log |Nx| + C_0 + O(|Nx|^{-2})}{(2/\pi) \log |1+k|^{1+\varepsilon/2} + C_0 + O(|1+k|^{-(2+\varepsilon)})} \end{aligned}$$

Now, since $|x| \leqslant k/N$, this last line is bounded above by a sequence tending to $(1+\varepsilon/2)^{-1}$ as $k \to \infty$. Thus, $P_x\{\tau \leqslant \tau_{D_N}\} \leqslant 1 - \alpha < 1$ for some $\alpha = \alpha(\varepsilon)$, independently of D and of k and of x such that $\delta(x) \leqslant k$.

Now, if $\delta(x) \leqslant k$ and $D^{(k)} = \{ z \in D_N \mid \delta(x) \leqslant k\}$, then

$$\begin{aligned} \sum_{y \in D^{(k)}} g_{D_N}(x, y) &= E_x\left\{ \sum_{n=0}^{\tau_{D_N}} 1_{D^{(k)}}(S_n) \right\} \\ &= E_x\left\{ \sum_{n=0}^{\tau_{D_N}} 1_{D^{(k)}}(S_n);\, \tau_{D_N} \leqslant \tau \right\} \\ &\quad + E_x\left\{ \sum_{n=0}^{\tau_{D_N}} 1_{D^{(k)}}(S_n),\, \tau_{D_N} > \tau \right\} \\ &= \mathrm{I} + \mathrm{II} \end{aligned}$$

If $\sigma = \min\{j \geqslant \tau \mid S_j \in D^{(k)}\}$, then by using the Markov property at time σ, we find

$$\begin{aligned}
\mathrm{II} &= E_x\left\{\tau_{D_N} > \tau,\ E_{S(\sigma)}\left\{\sum_{n=0}^{\tau_{D_N}} 1_{D^{(k)}}(S_n)\right\}\right\} \\
&\leqslant P_x\{\tau_{D_N} > \tau\} \sup_{v \in D^{(k)}} E_v\left\{\sum_{n=0}^{\tau_{D_N}} 1_{D^{(k)}}(S_n)\right\} \\
&\leqslant (1-\alpha) \sup_{v \in D^{(k)}} \sum_{y \in D^{(k)}} g_{D_N}(v, y)
\end{aligned}$$

On the other hand, let v be a lattice point outside $B(x)$ but one lattice step from it. Clearly $\tau \leqslant \tau_v = \min\{j \geqslant 0 \mid S_j = v\}$, so that

$$\begin{aligned}
\mathrm{I} &\leqslant E_x\left\{\sum_{n=0}^{\tau_v} 1_{B(x)}(S_n)\right\} \\
&= \sum_{y \in B(x)} E_x\left\{\sum_{n=0}^{\tau_v} 1_{\{y\}}(S_n)\right\} \\
&\leqslant |B(x)| \max_{y \in B(x)} 2a[N(y-v)] \\
&\leqslant C(1+k)^{2+\varepsilon} \log(1+k)
\end{aligned}$$

(Sections 10 and 11 of ref. 20 may be of help at the third step.) Thus, if

$$G = \sup_{x \in D^{(k)}} \sum_{y \in D^{(k)}} g_{D_N}(x, y)$$

and K is the right-hand side above, we find $G \leqslant K + (1-\alpha)G$, so that $G \leqslant K\alpha^{-1}$. Finally, if $x \in D_N \backslash D^{(k)}$ and $y \in D^{(k)}$, then $g_D(x, y) = E_x\, g_{D_N}(S_\sigma, y)$, hence

$$\sup_{x \in D_N} \sum_{y \in D^{(k)}} g_{D_N}(x, y) \leqslant G$$

and so the lemma is proved. ∎

Proof of Theorem A. We have

$$\begin{aligned}
&|u(x) - u_N(x)| \\
&\quad \leqslant |u(x) - E_x u(S(\tau_{D_N}))| + |E_x u(S(\tau_{D_N})) - E_x \phi(S(\tau_{D_N}))| \\
&\quad = \mathrm{I} + \mathrm{II}
\end{aligned}$$

Now, if $x \in \partial_N D$, then there exists $y \in \partial D$ with $|x-y| \leqslant 1/N$. By Lemma A.1,

$$\begin{aligned}
|u(x)-\phi(x)| &\leqslant |u(x)-\phi(y)| + |\phi(y)-\phi(x)| \\
&\leqslant C(\|\phi\|_\infty + \|\nabla\phi\|_\infty)\, |x-y|^{1/3} + \|\nabla\phi\|_\infty\, |x-y| \\
&\leqslant C(\phi)\, N^{-1/3}
\end{aligned}$$

where $C(\phi)$ is a short-hand for $C(\|\phi\|_\infty + \|\nabla\phi\|_\infty)$ which we will use here and below. It follows that $\mathrm{II} \leqslant C(\phi)\, N^{-1/3}$ as well.

Concerning I, note that

$$E_x u(S(\tau_{D_N})) - u(x) = \sum_{y \in D_N^0} g_{D_N}(x, y)\, \Delta_N u(y)$$

But by Lemmas A.2 and A.3 and summation by parts,

$$\begin{aligned}
&\sum_{\substack{y \in D_N^0 \\ \delta(y) \leqslant N^\alpha}} g_{D_N}(x, y)\, \Delta_N u(y) \\
&\quad = \sum_{k \leqslant N^\alpha} \sum_{\substack{y \in D_N^0 \\ \delta(y) = k}} g_{D_N}(x, y)\, \Delta_N u(y) \\
&\quad \leqslant C(\phi) \sum_{k \leqslant N^\alpha} \left(\frac{k+2}{N}\right)^{1/3} \sum_{\substack{y \in D_N^0 \\ \delta(y) = k}} g_{D_N}(x, y) \\
&\quad = C(\phi) \left\{ \left(\frac{[N^\alpha]+3}{N}\right)^{1/3} \sum_{\substack{y \in D_N^0 \\ \delta(y) \leqslant N^\alpha}} g_{D_N}(x, y) \right. \\
&\qquad \left. + \sum_{k \leqslant N^\alpha} \left[\left(\frac{k+2}{N}\right)^{1/3} - \left(\frac{k+3}{N}\right)^{1/3} \right] \sum_{y \in D^{(k)}} g_{D_N}(x, y) \right\} \\
&\quad \leqslant C(\phi) \left[N^{(\alpha-1)/3} N^{(2+\varepsilon)\alpha} \log N \right. \\
&\qquad \left. + \frac{1}{N^{1/3}} \sum_{k \leqslant N^\alpha} (k+3)^{-2/3}\, k^{2+\varepsilon} \log(1+k) \right] \\
&\quad \leqslant C(\phi)\, N^{[(7+3\varepsilon)\alpha - 1]/3} \log N
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{\substack{y \in D_N^0 \\ \delta(y) > N^\alpha}} g_{D_N}(x, y)\, \Delta_N u(y) \\
&\quad \leqslant C(\phi) \sum_{k \geqslant N^\alpha} k^{-3} \sum_{\substack{y \in D_N^0 \\ \delta(y) = k}} g_{D_N}(x, y) \\
&\quad \leqslant C(\phi) \left\{ \frac{-1}{([N^\alpha]+1)^3} \sum_{\delta(y) \leqslant N^\alpha} g_{D_N}(x, y) \right. \\
&\qquad \left. + \sum_{k \geqslant N^\alpha} [k^{-3} - (k+1)^{-3}] \sum_{y \in D^{(k)}} g_{D_N}(x, y) \right\} \\
&\quad \leqslant C(\phi) \left[N^{-3\alpha} N^{(2+\varepsilon)\alpha} \log N + \sum_{k \geqslant N^\alpha} k^{-4} k^{2+\varepsilon} \log k \right] \\
&\quad \leqslant C(\phi)\, N^{(\varepsilon - 1)\alpha} \log N
\end{aligned}$$

These two bounds are of the same order of magnitude when $3\alpha(\varepsilon - 1) = (7 + 3\varepsilon)\alpha - 1$, i.e., $\alpha = 1/10$, and that order of magnitude is $N^{-(1-\varepsilon)/10} \log N$. Fixing ε at slightly less than 1/11, to accommodate the logarithmic factor, finishes the proof. ∎

ACKNOWLEDGMENTS

This research was supported in part by an NSERC-Canada operating grant and in part by NSF grant NSF-DMS-89-3474.

This paper was substantially improved by remarks of a referee, who pointed out to us in particular the result in Proposition 3.8. In addition, the original version of this paper lacked a crucial estimate of the uniform approximation of classical harmonic measure by discrete harmonic measure. A suitable estimate was subsequently supplied to us by H. Kesten and it appears in the Appendix with his permission. It is a pleasure to thank both H. Kesten and the referee for their help.

REFERENCES

1. D. Bensimon *et al.*, Viscous flows in two dimensions, *Rev. Mod. Phys.* **58**:977–990 (1986).
2. C. Caratheodory, *Conformal Representation*, 2nd ed. (Cambridge University Press, Cambridge, 1958).
3. A. De Masi, P. A. Ferrari, and J. L. Lebowitz, Reaction-diffusion equations for interacting particle systems, *J. Stat. Phys.* **44**:589–644 (1986).

4. A. De Masi, P. A. Ferrari, and M. E. Vares, A microscopic model of interface related to the Burger's equation, *J. Stat. Phys.* **55**:601–611 (1989).
5. E. Di Benedetto and A. Friedman, The ill-posed Hele-Shaw model and the Stefan problem for supercooled water, *Trans. Am. Math. Soc.* **282**:183–204 (1984).
6. J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart* (Springer-Verlag, New York, 1984).
7. A. Friedman, *Variational Problems and Free Boundary Problems* (Wiley, New York, 1982).
8. A. Joffe and M. Metivier, Weak convergence of sequences of semimartingales with applications to multiple branching processes, *Adv. Appl. Prob.* **18**:20–65 (1986).
9. L. Kadanoff, Simulating hydrodynamics: A pedestrian model, *J. Stat. Phys.* **39**:267–283 (1985).
10. O. D. Kellogg, *Foundations of Potential Theory* (Verlag von J. Springer, Berlin, 1929).
11. H. Kesten, How long are the arms in DLA?, *J. Phys. A: Math. Gen.* **20**:L29–L33 (1987).
12. H. Kesten, Hitting probabilities of random walks on $\mathbb{Z}^d$, *Stoch. Proc. Appl.* **25**:165–184 (1987).
13. H. Kesten, Some caricatures of multiple contact diffusion limited aggregation and the η-model, in *Proceedings of Durham Symposium on Stochastic Analysis*, to appear.
14. C. Kipnis, S. Olla, and S. R. S. Varadhan, Hydrodynamics and large deviation for simple exclusion processes, *Commun. Pure Appl. Math.* **42**:115–137 (1989).
15. S. Liang, Random walk simulations of flow in Hele-Shaw cells, *Phys. Rev. A* **33**:2663–2674 (1986).
16. I. Mitoma, Tightness of probabilities on $C([0, 1], \mathscr{S}')$ and $D([0, 1], \mathscr{S}')$, *Ann. Prob.* **11**:989–999 (1983).
17. C. Pommerenke, *Univalent Functions* (Vanderhoeck and Ruprecht, 1976).
18. L. Paterson, Diffusion limited aggregation and two fluid displacement in porous media, *Phys. Rev. Lett.* **53**:1621–1624 (1984).
19. M. Plischke, Z. Rácz, and D. Liu, Time reversal invariance and universality of two dimensional growth models, *Phys. Rev. B* **35**:3485–3495 (1987).
20. F. Spitzer, *Principles of Random Walk* (Van Nostrand, New York).
21. H. E. Stanley and N. Ostrowsky, *On Growth and Form: Fractal and Non-fractal Patterns in Physics* (M. Nijhoff, Boston, 1986).
22. H. E. Stanley and N. Ostrowsky, *Nato Advanced Study Institute on Random Fluctuations and Pattern Growth: Experiments and Models* (Kluwer Academic Publishers, Boston, 1988).
23. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, New Jersey, 1970).
24. D. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes* (Springer-Verlag, Berlin, 1979).
25. J. Szep, J. Cserti, and J. Kertész, Monte-Carlo approach to dendritic growth, *J. Phys. A* **18**:L413–418 (1985).
26. T. Viczek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).
27. T. A. Witten and L. M. Sander, Diffusion limited aggregation, a kinetic critical phenomenon, *Phys. Rev. Lett.* **47**:1400–1403 (1981).